GLOBAL EDITION

A First Course in Probability

TENTH EDITION

Sheldon Ross

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Tenth Edition

Global Edition

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Pearson

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Pearson Education Limited
KAO Two
KAO Park
Harlow
CM17 9SR
United Kingdom

and Associated Companies throughout the world

Visit us on the World Wide Web at: www.pearsonglobaleditions.com

ISBN 10: 1-292-26920-0
ISBN 13: 978-1-292-26920-7
ebook ISBN 13: 978-1-292-26923-8

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library

For Rebecca

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Preface

"We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it. . . . It is remarkable that this science, which originated in the consideration of games of chance, should have become the most important object of human knowledge. . . . The most important questions of life are, for the most part, really only problems of probability." So said the famous French mathematician and astronomer (the "Newton of France") Pierre-Simon, Marquis de Laplace. Although many people believe that the famous marquis, who was also one of the great contributors to the development of probability, might have exaggerated somewhat, it is nevertheless true that probability theory has become a tool of fundamental importance to nearly all scientists, engineers, medical practitioners, jurists, and industrialists. In fact, the enlightened individual had learned to ask not "Is it so?" but rather "What is the probability that it is so?"

General Approach and Mathematical Level

This book is intended as an elementary introduction to the theory of probability for students in mathematics, statistics, engineering, and the sciences (including computer science, biology, the social sciences, and management science) who possess the prerequisite knowledge of elementary calculus. It attempts to present not only the mathematics of probability theory, but also, through numerous examples, the many diverse possible applications of this subject.

Content and Course Planning

Chapter 1 presents the basic principles of combinatorial analysis, which are most useful in computing probabilities.

Chapter 2 handles the axioms of probability theory and shows how they can be applied to compute various probabilities of interest.

Chapter 3 deals with the extremely important subjects of conditional probability and independence of events. By a series of examples, we illustrate how conditional probabilities come into play not only when some partial information is available, but also as a tool to enable us to compute probabilities more easily, even when no partial information is present. This extremely important technique of obtaining probabilities by "conditioning" reappears in Chapter 7, where we use it to obtain expectations.

The concept of random variables is introduced in Chapters 4, 5, and 6. Discrete random variables are dealt with in Chapter 4, continuous random variables in Chapter 5, and jointly distributed random variables in Chapter 6. The important concepts of the expected value and the variance of a random variable are introduced in Chapters 4 and 5, and these quantities are then determined for many of the common types of random variables.

Additional properties of the expected value are considered in Chapter 7. Many examples illustrating the usefulness of the result that the expected value of a sum of random variables is equal to the sum of their expected values are presented. Sections on conditional expectation, including its use in prediction, and on moment-generating functions are contained in this chapter. In addition, the final section introduces the multivariate normal distribution and presents a simple proof concerning the joint distribution of the sample mean and sample variance of a sample from a normal distribution.

Chapter 8 presents the major theoretical results of probability theory. In particular, we prove the strong law of large numbers and the central limit theorem. Our proof of the strong law is a relatively simple one that assumes that the random variables have a finite fourth moment, and our proof of the central limit theorem assumes Levy's continuity theorem. This chapter also presents such probability inequalities as Markov's inequality, Chebyshev's inequality, and Chernoff bounds. The final section of Chapter 8 gives a bound on the error involved when a probability concerning a sum of independent Bernoulli random variables is approximated by the corresponding probability of a Poisson random variable having the same expected value.

Chapter 9 presents some additional topics, such as Markov chains, the Poisson process, and an introduction to information and coding theory, and Chapter 10 considers simulation.

As in the previous edition, three sets of exercises are given at the end of each chapter. They are designated as **Problems, Theoretical Exercises**, and **Self-Test Problems and Exercises**. This last set of exercises, for which complete solutions appear in Solutions to Self-Test Problems and Exercises, is designed to help students test their comprehension and study for exams.

Changes for the Tenth Edition

The tenth edition continues the evolution and fine tuning of the text. Aside from a multitude of small changes made to increase the clarity of the text, the new edition includes many new and updated problems, exercises, and text material chosen both for inherent interest and for their use in building student intuition about probability. Illustrative of these goals are Examples 4n of Chapter 3, which deals with computing NCAA basketball tournament win probabilities, and Example 5b of Chapter 4, which introduces the friendship paradox. There is also new material on the Pareto distribution (introduced in Section 5.6.5), on Poisson limit results (in Section 8.5), and on the Lorenz curve (in Section 8.7).

Acknowledgments

I would like to thank the following people who have graciously taken the time to contact me with comments for improving the text: Amir Ardestani, Polytechnic University of Teheran; Joe Blitzstein, Harvard University; Peter Nuesch, University of Lausaunne; Joseph Mitchell, SUNY, Stony Brook; Alan Chambless, actuary; Robert Kriner; Israel David, Ben-Gurion University; T. Lim, George Mason University; Wei Chen, Rutgers; D. Monrad, University of Illinois; W. Rosenberger, George Mason University; E. Ionides, University of Michigan; J. Corvino, Lafayette College;

T. Seppalainen, University of Wisconsin; Jack Goldberg; University of Michigan; Sunil Dhar, New Jersey Institute of Technology; Vladislav Kargin, Stanford University; Marlene Miller; Ahmad Parsian; and Fritz Scholz, University of Washington.

I would also like to especially thank the reviewers of the ninth and tenth editions: Richard Laugesen, University of Illinois; Stacey Hancock, Clark University; Stefan Heinz, University of Wyoming; and Brian Thelen, University of Michigan; Mark Ward, Purdue University. I would like to thank the accuracy checker, Stacey Hancock (Montana State University), for her careful review.

Finally, I would like to thank the following reviewers for their many helpful comments. Reviewers of the tenth edition are marked with an asterisk.

K. B. Athreya, *Iowa State University*
Richard Bass, *University of Connecticut*
Robert Bauer, *University of Illinois at Urbana-Champaign*
Phillip Beckwith, *Michigan Tech*
Arthur Benjamin, *Harvey Mudd College*
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R. Miller, *Stanford University*
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Acknowledgments for the Global Edition

Pearson would like to thank and acknowledge the following people for their contributions to the Global Edition.

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Chapter 1 Combinatorial Analysis

Contents

1.1 Introduction

Here is a typical problem of interest involving probability: A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals—and will be called *functional*—as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where $n = 4$ and $m = 2$, there are 6 possible system configurations, namely,

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

where 1 means that the antenna is working and 0 that it is defective. Because the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability. In the case of general n and m, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system's being functional and then divide by the total number of all possible configurations.

From the preceding discussion, we see that it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. The mathematical theory of counting is formally known as *combinatorial analysis*.

1.2 The Basic Principle of Counting

The basic principle of counting will be fundamental to all our work. Loosely put, it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are mn possible outcomes of the two experiments.

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Proof of the Basic Principle: The basic principle may be proven by enumerating all the possible outcomes of the two experiments; that is,

$$\begin{array}{l}(1,1),\ (1,2),\ \ldots,\ (1,n)\\(2,1),\ (2,2),\ \ldots,\ (2,n)\\\ \vdots\\(m,1),\ (m,2),\ \ldots,\ (m,n)\end{array}$$

where we say that the outcome is (i, j) if experiment 1 results in its ith possible outcome and experiment 2 then results in its jth possible outcome. Hence, the set of possible outcomes consists of m rows, each containing n elements. This proves the result.

Example 2a

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices. ■

When there are more than two experiments to be performed, the basic principle can be generalized.

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 2b A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees. ■

Example 2c How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution By the generalized version of the basic principle, the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$. ■

Example 2d How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions. ■

Example 2e In Example 2c, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution In this case, there would be $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$ possible license plates. ■

1.3 Permutations

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6, namely, *abc, acb, bac, bca, cab*, and *cba*. Each arrangement is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

$$n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Whereas $n!$ (read as "n factorial") is defined to equal $1 \cdot 2 \cdots n$ when n is a positive integer, it is convenient to define 0! to equal 1.

Example 3a How many different batting orders are possible for a baseball team consisting of 9 players?

Solution There are $9! = 362{,}880$ possible batting orders. ■

Example 3b A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?

(b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Solution (a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3{,}628{,}800$.

(b) Since there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in this case. ■

Example 3c Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is $4!\,4!\,3!\,2!\,1! = 6912$. ■

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from one another. To set this situation straight in our minds, consider the following example.

Example 3d How many different letter arrangements can be formed from the letters *PEPPER*?

Solution We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from one another. However, consider any one of these permutations—for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form *PPEPER*. That is, all 3! 2! permutations

$$
\begin{array}{ll}
P_1P_2E_1P_3E_2R & P_1P_2E_2P_3E_1R \\
P_1P_3E_1P_2E_2R & P_1P_3E_2P_2E_1R \\
P_2P_1E_1P_3E_2R & P_2P_1E_2P_3E_1R \\
P_2P_3E_1P_1E_2R & P_2P_3E_2P_1E_1R \\
P_3P_1E_1P_2E_2R & P_3P_1E_2P_2E_1R \\
P_3P_2E_1P_1E_2R & P_3P_2E_2P_1E_1R
\end{array}
$$

are of the form *PPEPER*. Hence, there are $6!/(3!\,2!) = 60$ possible letter arrangements of the letters *PEPPER*. ■

In general, the same reasoning as that used in Example 3d shows that there are

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike.

Example 3e A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution There are

$$\frac{10!}{4!\,3!\,2!\,1!} = 12{,}600$$

possible outcomes. ■

Example 3f How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution There are

$$\frac{9!}{4!\,3!\,2!} = 1260$$

different signals. ■

1.4 Combinations

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this question, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3—say, the group consisting of items A, B, and C—will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, and CBA will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ (read as "n choose r") represents the number of possible combinations of n objects taken r at a time.

Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

Equivalently, $\binom{n}{r}$ is the number of subsets of size r that can be chosen from a set of size n. Using that $0! = 1$, note that $\binom{n}{n} = \binom{n}{0} = \frac{n!}{0!n!} = 1$, which is consistent with the preceding interpretation because in a set of size n there is exactly 1 subset of size n (namely, the entire set), and exactly one subset of size 0 (namely the empty set). A useful convention is to define $\binom{n}{r}$ equal to 0 when either $r > n$ or $r < 0$.

Example 4a

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees. ■

Example 4b

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$ possible committees consisting of 2 women and 3 men.

Now suppose that 2 of the men refuse to serve together. Because a total of $\binom{2}{2}\binom{5}{1} = 5$ out of the $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ groups that do not contain both of the feuding men. Because there are still $\binom{5}{2} = 10$ ways to choose the 2 women, there are $30 \cdot 10 = 300$ possible committees in this case. ■

Example 4c

Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the

$\wedge\, 1 \,\wedge\, 1 \,\wedge\, 1 \ldots \wedge\, 1 \,\wedge\, 1 \,\wedge$

1 = functional

$\wedge$ = place for at most one defective

Figure 1.1 No consecutive defectives.

functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions—represented in Figure 1.1 by carets—between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones. ■

A useful combinatorial identity, known as *Pascal's identity*, is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \le r \le n \tag{4.1}$$

Equation (4.1) may be proved analytically or by the following combinatorial argument: Consider a group of n objects, and fix attention on some particular one of these objects—call it object 1. Now, there are $\binom{n-1}{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r - 1$ from the remaining $n - 1$ objects). Also, there are $\binom{n-1}{r}$ groups of size r that do not contain object 1. As there is a total of $\binom{n}{r}$ groups of size r, Equation (4.1) follows.

The values $\binom{n}{r}$ are often referred to as *binomial coefficients* because of their prominence in the binomial theorem.

The binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4.2}$$

We shall present two proofs of the binomial theorem. The first is a proof by mathematical induction, and the second is a proof based on combinatorial considerations.

Proof of the Binomial Theorem by Induction: When $n = 1$, Equation (4.2) reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x$$

Assume Equation (4.2) for $n - 1$. Now,

$$\begin{aligned}
(x + y)^n &= (x + y)(x + y)^{n-1} \\
&= (x + y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}
\end{aligned}$$

Letting $i = k + 1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}
(x + y)^n &= \sum_{i=1}^{n}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=0}^{n-1}\binom{n-1}{i}x^i y^{n-i} \\
&= \sum_{i=1}^{n-1}\binom{n-1}{i-1}x^i y^{n-i} + x^n + y^n + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1}\left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}
\end{aligned}$$

where the next-to-last equality follows by Equation (4.1). By induction, the theorem is now proved.

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2)\cdots(x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, \ldots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have k of the x_i's and $(n - k)$ of the y_i's as factors? As each term consisting of k of the x_i's and $(n - k)$ of the y_i's corresponds to a choice of a group of k from the n values $x_1, x_2, \ldots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x, y_i = y, i = 1, \ldots, n$, we see that

$$(x + y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

Example 4d Expand $(x + y)^3$.

Solution

$$(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0$$
$$= y^3 + 3xy^2 + 3x^2y + x^3$$ ■

Example 4e How many subsets are there of a set consisting of n elements?

Solution Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1 + 1)^n = 2^n$$

This result could also have been obtained by assigning either the number 0 or the number 1 to each element in the set. To each assignment of numbers, there corresponds, in a one-to-one fashion, a subset, namely, that subset consisting of all elements that were assigned the value 1. As there are 2^n possible assignments, the result follows.

Note that we have included the set consisting of 0 elements (that is, the null set) as a subset of the original set. Hence, the number of subsets that contain at least 1 element is $2^n - 1$. ■

1.5 Multinomial Coefficients

In this section, we consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible? To answer this question, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n - n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n - n_1 - n_2}{n_3}$ possible choices for the third group; and so on. It then follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n - n_1}{n_2}\cdots\binom{n - n_1 - n_2 - \cdots - n_{r-1}}{n_r}$$
$$= \frac{n!}{(n - n_1)!\, n_1!} \frac{(n - n_1)!}{(n - n_1 - n_2)!\, n_2!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{r-1})!}{0!\, n_r!}$$
$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

possible divisions.

Another way to see this result is to consider the n values $1, 1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r$, where i appears n_i times, for $i = 1, \ldots, r$. Every permutation of these values corresponds to a division of the n items into the r groups in the following manner: Let the permutation $i_1, i_2, \ldots, i_n$ correspond to assigning item 1 to group i_1, item 2 to group i_2, and so on. For instance, if $n = 8$ and if $n_1 = 4$, $n_2 = 3$, and $n_3 = 1$, then the permutation $1, 1, 2, 3, 2, 1, 2, 1$ corresponds to assigning items $1, 2, 6, 8$ to the first group, items $3, 5, 7$ to the second group, and item 4 to the third group. Because every permutation yields a division of the items and every possible division results from some permutation, it follows that the number of divisions of n items into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ is the same as the number of permutations of n items of which n_1 are alike, and n_2 are alike, ..., and n_r are alike, which was shown in Section 1.3 to equal $\dfrac{n!}{n_1!n_2!\cdots n_r!}$.

Notation

If $n_1 + n_2 + \cdots + n_r = n$, we define $\dbinom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Thus, $\dbinom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 5a

A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution There are $\dfrac{10!}{5!\,2!\,3!} = 2520$ possible divisions. ■

Example 5b

Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution There are $\dfrac{10!}{5!\,5!} = 252$ possible divisions. ■

Example 5c

In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution Note that this example is different from Example 5b because now the order of the two teams is irrelevant. That is, there is no A or B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is

$$\frac{10!/(5!\,5!)}{2!} = 126$$ ■

The proof of the following theorem, which generalizes the binomial theorem, is left as an exercise.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r): \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients*.

Example 5d

In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players.

(a) How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)

(b) How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?

Solution One way to determine the number of possible outcomes for the initial round is to first determine the number of possible pairings for that round. To do so, note that the number of ways to divide the 8 players into a *first* pair, a *second* pair, a *third* pair, and a *fourth* pair is $\binom{8}{2,2,2,2} = \frac{8!}{2^4}$. Thus, the number of possible pairings when there is no ordering of the 4 pairs is $\frac{8!}{2^4\,4!}$. For each such pairing, there are 2 possible choices from each pair as to the winner of that game, showing that there are $\frac{8!2^4}{2^4\,4!} = \frac{8!}{4!}$ possible results of round 1. [Another way to see this is to note that there are $\binom{8}{4}$ possible choices of the 4 winners and, for each such choice, there are 4! ways to pair the 4 winners with the 4 losers, showing that there are $4!\binom{8}{4} = \frac{8!}{4!}$ possible results for the first round.]

Similarly, for each result of round 1, there are $\frac{4!}{2!}$ possible outcomes of round 2, and for each of the outcomes of the first two rounds, there are $\frac{2!}{1!}$ possible outcomes of round 3. Consequently, by the generalized basic principle of counting, there are $\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8!$ possible outcomes of the tournament. Indeed, the same argument can be used to show that a knockout tournament of $n = 2^m$ players has $n!$ possible outcomes.

Knowing the preceding result, it is not difficult to come up with a more direct argument by showing that there is a one-to-one correspondence between the set of possible tournament results and the set of permutations of $1, \ldots, n$. To obtain such a correspondence, rank the players as follows for any tournament result: Give the tournament winner rank 1, and give the final-round loser rank 2. For the two players who lost in the next-to-last round, give rank 3 to the one who lost to the player ranked 1 and give rank 4 to the one who lost to the player ranked 2. For the four players who lost in the second-to-last round, give rank 5 to the one who lost to player ranked 1, rank 6 to the one who lost to the player ranked 2, rank 7 to the one who lost to the player ranked 3, and rank 8 to the one who lost to the player ranked 4. Continuing on in this manner gives a rank to each player. (A more succinct description is to give the winner of the tournament rank 1 and let the rank of a player who lost in a round having 2^k matches be 2^k plus the rank of the player who beat him, for $k = 0, \ldots, m - 1$.) In this manner, the result of the tournament can be represented by a permutation $i_1, i_2, \ldots, i_n$, where i_j is the player who was given rank j. Because different tournament results give rise to different permutations, and because there is a tournament result for each permutation, it follows that there are the same number of possible tournament results as there are permutations of $1, \ldots, n$. ■

Example 5e

$$
\begin{aligned}
(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 \\
&\quad + \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 \\
&\quad + \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\
&= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3
\end{aligned}
$$
■

*1.6 The Number of Integer Solutions of Equations

An individual has gone fishing at Lake Ticonderoga, which contains four types of fish: lake trout, catfish, bass, and bluefish. If we take the result of the fishing trip to be the numbers of each type of fish caught, let us determine the number of possible outcomes when a total of 10 fish are caught. To do so, note that we can denote the outcome of the fishing trip by the vector (x_1, x_2, x_3, x_4) where x_1 is the number of trout that are caught, x_2 is the number of catfish, x_3 is the number of bass, and x_4 is the number of bluefish. Thus, the number of possible outcomes when a total of 10 fish are caught is the number of nonnegative integer vectors (x_1, x_2, x_3, x_4) that sum to 10.

More generally, if we supposed there were r types of fish and that a total of n were caught, then the number of possible outcomes would be the number of nonnegative integer-valued vectors $x_1, \ldots, x_r$ such that

$$x_1 + x_2 + \ldots + x_r = n \tag{6.1}$$

To compute this number, let us start by considering the number of positive integer-valued vectors $x_1, \ldots, x_r$ that satisfy the preceding. To determine this number, suppose that we have n consecutive zeroes lined up in a row:

$$0\,0\,0 \ldots 0\,0$$

* Asterisks denote material that is optional.

$0_\wedge 0_\wedge 0_\wedge \ldots _\wedge 0_\wedge 0$

n objects 0

Choose $r - 1$ of the spaces $\wedge$.

Figure 1.2 Number of positive solutions.

Note that any selection of $r - 1$ of the $n - 1$ spaces between adjacent zeroes (see Figure 1.2) corresponds to a positive solution of (6.1) by letting x_1 be the number of zeroes before the first chosen space, x_2 be the number of zeroes between the first and second chosen space, ..., and x_n being the number of zeroes following the last chosen space.
For instance, if we have $n = 8$ and $r = 3$, then (with the choices represented by dots) the choice

$$0 . 0\,0\,0\,0 . 0\,0\,0$$

corresponds to the solution $x_1 = 1, x_2 = 4, x_3 = 3$. As positive solutions of (6.1) correspond, in a one-to-one fashion, to choices of $r - 1$ of the adjacent spaces, it follows that the number of differerent positive solutions is equal to the number of different selections of $r - 1$ of the $n - 1$ adjacent spaces. Consequently, we have the following proposition.

Proposition 6.1

There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n, \quad x_i > 0, \quad i = 1, \ldots, r$$

To obtain the number of nonnegative (as opposed to positive) solutions, note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive solutions of $y_1 + \cdots + y_r = n + r$ (seen by letting $y_i = x_i + 1,\ i = 1, \ldots, r$). Hence, from Proposition 6.1, we obtain the following proposition.

Proposition 6.2

There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n$$

Thus, using Proposition 6.2, we see that there are $\binom{13}{3} = 286$ possible outcomes when a total of 10 Lake Ticonderoga fish are caught.

Example 6a

How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

Solution There are $\binom{3+2-1}{2-1} = 4$ such solutions: (0, 3), (1, 2), (2, 1), (3, 0). ■

Example 6b

An investor has \$20,000 to invest among 4 possible investments. Each investment must be in units of \$1000. If the total \$20,000 is to be invested, how many different investment strategies are possible? What if not all the money needs to be invested?

Solution If we let $x_i,\ i = 1, 2, 3, 4$, denote the number of thousands invested in investment i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \quad x_i \geq 0$$

Hence, by Proposition 6.2, there are $\binom{23}{3} = 1771$ possible investment strategies. If not all of the money needs to be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector $(x_1, x_2, x_3, x_4, x_5)$ satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

Hence, by Proposition 6.2, there are now $\binom{24}{4} = 10{,}626$ possible strategies. ■

Example 6c

How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$?

Solution

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all nonnegative integer-valued $(n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_r = n$. Hence, by Proposition 6.2, there are $\binom{n+r-1}{r-1}$ such terms. ■

Example 6d

Let us consider again Example 4c, in which we have a set of n items, of which m are (indistinguishable and) defective and the remaining $n - m$ are (also indistinguishable and) functional. Our objective is to determine the number of linear orderings in which no two defectives are next to each other. To determine this number, let us imagine that the defective items are lined up among themselves and the functional ones are now to be put in position. Let us denote x_1 as the number of functional items to the left of the first defective, x_2 as the number of functional items between the first two defectives, and so on. That is, schematically, we have

$$x_1\ 0\ x_2\ 0 \cdots x_m\ 0\ x_{m+1}$$

Now, there will be at least one functional item between any pair of defectives as long as $x_i > 0,\ i = 2, \ldots, m$. Hence, the number of outcomes satisfying the condition is the number of vectors $x_1, \ldots, x_{m+1}$ that satisfy the equation

$$x_1 + \cdots + x_{m+1} = n - m, \quad x_1 \geq 0,\ x_{m+1} \geq 0,\ x_i > 0,\ i = 2, \ldots, m$$

But, on letting $y_1 = x_1 + 1, y_i = x_i,\ i = 2, \ldots, m,\ y_{m+1} = x_{m+1} + 1$, we see that this number is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy the equation

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2$$

Hence, by Proposition 6.1, there are $\binom{n - m + 1}{m}$ such outcomes, in agreement with the results of Example 4c.

Suppose now that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional items. By the same reasoning as that applied previously, this would equal the number of vectors satisfying the equation

$$x_1 + \cdots + x_{m+1} = n - m, \quad x_1 \geq 0,\ x_{m+1} \geq 0,\ x_i \geq 2,\ i = 2, \ldots, m$$

Upon letting $y_1 = x_1 + 1, y_i = x_i - 1, i = 2, \ldots, m, y_{m+1} = x_{m+1} + 1$, we see that this is the same as the number of positive solutions of the equation

$$y_1 + \cdots + y_{m+1} = n - 2m + 3$$

Hence, from Proposition 6.1, there are $\binom{n - 2m + 2}{m}$ such outcomes. ■

Summary

The basic principle of counting states that if an experiment consisting of two phases is such that there are n possible outcomes of phase 1 and, for each of these n outcomes, there are m possible outcomes of phase 2, then there are nm possible outcomes of the experiment.

There are $n! = n(n - 1) \cdots 3 \cdot 2 \cdot 1$ possible linear orderings of n items. The quantity 0! is defined to equal 1.

Let

$$\binom{n}{i} = \frac{n!}{(n - i)!\, i!}$$

when $0 \leq i \leq n$, and let it equal 0 otherwise. This quantity represents the number of different subgroups of size i that can be chosen from a set of size n. It is often called a *binomial coefficient* because of its prominence in the binomial theorem, which states that

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

For nonnegative integers $n_1, \ldots, n_r$ summing to n,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

is the number of divisions of n items into r distinct nonoverlapping subgroups of sizes $n_1, n_2 \ldots, n_r$. These quantities are called *multinomial coefficients*.

Problems

1. (a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?

(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

3. Ten employees of a company are to be assigned to 10 different managerial posts, one to each post. In how many ways can these posts be filled?

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

5. A safe can be opened by inserting a code consisting of three digits between 0 and 9. How many codes are possible? How many codes are possible with no digit repeated? How many codes starting with a 1 are possible?

6. A well-known nursery rhyme starts as follows:

"As I was going to St. Ives
I met a man with 7 wives.
Each wife had 7 sacks.
Each sack had 7 cats.
Each cat had 7 kittens..."
How many kittens did the traveler meet?

7. (a) In how many ways can 3 boys and 3 girls sit in a row?

(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

(c) In how many ways if only the boys must sit together?

(d) In how many ways if no two people of the same sex are allowed to sit together?

8. When all letters are used, how many different letter arrangements can be made from the letters

(a) Partying?

(b) Dancing?

(c) Acting?

(d) Singing?

9. A box contains 13 balls, of which 4 are yellow, 4 are green, 3 are red, and 2 are blue. Find the number of ways in which these balls can be arranged in a line.

10. In how many ways can 8 people be seated in a row if

(a) there are no restrictions on the seating arrangement?

(b) persons A and B must sit next to each other?

(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

(d) there are 5 men and they must sit next to one another?

(e) there are 4 married couples and each couple must sit together?

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if

(a) the books can be arranged in any order?

(b) the mathematics books must be together and the novels must be together?

(c) the novels must be together, but the other books can be arranged in any order?

12. How many 3 digit numbers xyz, with x, y, z all ranging from 0 to 9 have at least 2 of their digits equal. How many have exactly 2 equal digits.

13. How many different letter configurations of length 4 or 5 can be formed using the letters of the word A C H I E V E?

14. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if

(a) a student can receive any number of awards?

(b) each student can receive at most 1 award?

15. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

16. How many distinct triangles can be drawn by joining any 8 dots on a piece of paper? Note that the dots are in such a way that no 3 of them form a straight line.

17. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

18. A team consisting of 5 players is to be chosen from a class of 12 boys and 9 girls. How many choices are possible if

(a) all players are of the same gender?

(b) the team includes both genders?

19. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

20. A team of 9, consisting of 2 mathematicians, 3 statisticians, and 4 physicists, is to be selected from a faculty of 10 mathematicians, 8 statisticians, and 7 physicists. How many teams are possible?

21. From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

(a) 2 of the men refuse to serve together?

(b) 2 of the women refuse to serve together?

(c) 1 man and 1 woman refuse to serve together?

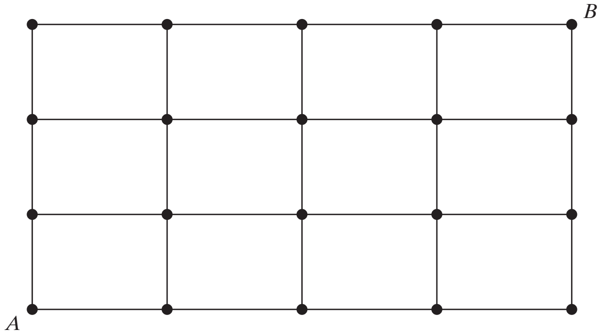
22. A person has 8 friends, of whom 5 will be invited to a party.

(a) How many choices are there if 2 of the friends are feuding and will not attend together?

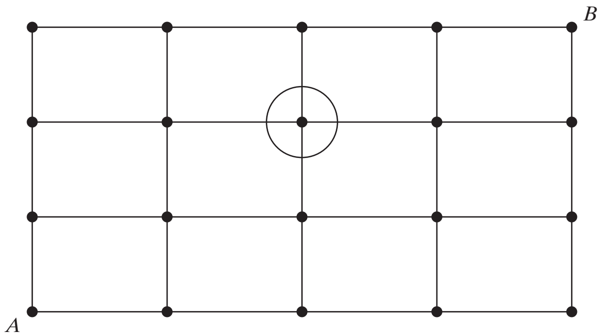
(b) How many choices if 2 of the friends will only attend together?

23. Consider the grid of points shown at the top of the next column. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

Hint: Note that to reach B from A, you must take 4 steps to the right and 3 steps upward.



24. In Problem 23, how many different paths are there from A to B that go through the point circled in the following lattice?



25. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

26. (a) Show $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$

(b) Simplify $\sum_{k=0}^{n} \binom{n}{k} x^k$

27. Expand $(4x - 3y)^4$.

28. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

29. Expand $(x_1 + 2x_2 + 3x_3)^4$.

30. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

31. If 10 gifts are to be distributed among 3 friends, how many distributions are possible? What if each friend should receive at least 3 gifts?

32. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

33. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other and the Russian and U.S. delegates are not to be next to each other?

***34.** If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?

***35.** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

***36.** We have $20,000 that must be invested among 4 possible opportunities. Each investment must be integral in units of $1000, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are $2000, $2000, $3000, and $4000. How many different investment strategies are available if

(a) an investment must be made in each opportunity?

(b) investments must be made in at least 3 of the 4 opportunities?

***37.** Suppose that 10 fish are caught at a lake that contains 5 distinct types of fish.

(a) How many different outcomes are possible, where an outcome specifies the numbers of caught fish of each of the 5 types?

(b) How many outcomes are possible when 3 of the 10 fish caught are trout?

(c) How many when at least 2 of the 10 are trout?

Theoretical Exercises

1. Prove the generalized version of the basic counting principle.

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n objects if the order of selection is considered relevant?

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n - r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of $k, k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n - k + 1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n - k + 1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

11. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.
Hint: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest numbered member?

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.
Hint:

(i) How many possible selections are there of a committee of size k and its chairperson?

(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n + 1)$$

For a combinatorial proof of the preceding, consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).